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Lemvig, Jakob

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The Canonical and Alternate Duals of a Wavelet Frame

Jakob Lemvig
[Joint with: Marcin Bownik, University of Oregon]

Department of Mathematics, Technical University of Denmark (DTU)

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Motivation.

Wavelet frames: A generalization of wavelet orthonormal basis with more flexibility and freedom.

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Wavelet frames: A generalization of wavelet orthonormal basis with more flexibility and freedom.

Do we always have this freedom?
The natural representation of a function in terms of a frame involves the so-called canonical dual. The canonical dual of a wavelet frame need not have wavelet structure; and even worse, there might not be any dual with wavelet structure.

One Result in this Direction

Suppose ψ is an orthonormal wavelet and let

$$\theta(x) = \psi(x) + \varepsilon 2^{1/2}\psi(2x)$$

for $0 < \varepsilon < 1$. Then θ generates a wavelet Riesz basis whose (canonical) dual is not of the form

$$\{2^{j/2}\phi(2^j x - k) \, : \, j, k \in \mathbb{Z}, \phi \in \Phi\}$$

for any finite set Φ of generators ([Daubechies], [Chui & Shi]).

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$$\{2^{j/2}\phi(2^j x - k) \, : \, j, k \in \mathbb{Z}, \phi \in \Phi\}$$

for any finite set Φ of generators ([Daubechies], [Chui & Shi]). Obviously, the Riesz wavelet has no alternate wavelet duals either, so one might ask:

Does the existence of a dual wavelet frame imply wavelet structure of the canonical dual?

Main result of this work

In this talk we will explore the relationship between canonical and alternate dual frames of a wavelet frame. The main result is a negative answer to the question on the previous slide:

Theorem (Bownik, L.)

For all $J \in \mathbb{N}$, there exists a frame wavelet $\psi \in L^2(\mathbb{R})$ such that:

(i) $\hat{\psi}$ is C^∞ and compactly supported,

(ii) its canonical dual frame is not a wavelet system generated by fewer than 2^J functions,

(iii) there are infinitely many $\tilde{\psi}$ such that ψ and $\tilde{\psi}$ form a pair of dual wavelet frames.

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(ii) *its canonical dual frame is not a wavelet system generated by fewer than 2^J functions,*

(iii) *there are infinitely many $\tilde{\psi}$ such that ψ and $\tilde{\psi}$ form a pair of dual wavelet frames.*

This claim (with $J = 1$) was asserted by Daubechies and Han [Appl. Comp. Harmonic Anal. **12** (2002), no. 3, 269–285].

What is a Frame?

A sequence $\{f_j\}_{j\in\mathbb{N}}$ is a frame for a separable Hilbert space $\mathcal{H}$ if

$$\exists C_1, C_2 > 0 : \quad C_1 \|f\|^2 \le \sum_{j\in\mathbb{N}} |\langle f, f_j\rangle|^2 \le C_2 \|f\|^2 \qquad \text{for all } f \in \mathcal{H}.$$

If the upper bound holds in the above inequality, then $\{f_j\}$ is said to be a Bessel sequence. Two Bessel sequences $\{f_j\}$ and $\{g_j\}$ are said to be dual frames if

$$f = \sum_{j\in\mathbb{N}} \langle f, g_j\rangle f_j \quad \text{for all } f \in \mathcal{H}.$$

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$$f = \sum_{j\in\mathbb{N}} \langle f, \mathrm{S}^{-1} f_j\rangle f_j \quad \text{for all } f \in \mathcal{H}.$$

At least one dual always exists: the canonical dual $\{\mathrm{S}^{-1} f_j\}$, where the frame operator of $\{f_j\}$ is $\mathrm{S} \colon \mathcal{H} \to \mathcal{H}, \mathrm{S} f = \sum \langle f, f_j\rangle f_j$. Redundant frames have several duals; a dual which is not the canonical dual is called an alternate dual.

What is a Frame Wavelet?

The wavelet system generated by $\Psi = \{\psi_1, \ldots, \psi_L\}$ is defined as

$\{\psi_{j,k} : j, k \in \mathbb{Z}, \psi \in \Psi\}$, where $\psi_{j,k} := \psi(a^j \cdot - k) \equiv D_a^j T_k \psi \equiv \mathcal{U}(\Psi)$.

where the translation and dilation operators on $L^2(\mathbb{R})$ are:

$$T_k f(x) := f(x - k), \quad k \in \mathbb{Z},$$
$$D_a f(x) := a^{1/2} f(ax), \quad a > 1.$$

The wavelet unitaries are denoted $\mathcal{U} := \{D_a^j T_k : j \in \mathbb{Z}, k \in \mathbb{Z}\}$.

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A set of functions $\Psi \in L^2(\mathbb{R})$ is called a frame wavelet if $\{\psi_{j,k}\}_{j\in\mathbb{Z}, k\in\mathbb{Z}, \psi\in\Psi}$ is a frame for $L^2(\mathbb{R})$.

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Expansion property

Suppose Ψ is a frame wavelet. Then

$$f = \sum_{\psi \in \Psi} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \langle f, S^{-1}\psi_{j,k} \rangle \psi_{j,k} \quad \text{for all } f \in L^2(\mathbb{R}),$$

Notation

A closed subspace $W \subset L^2(\mathbb{R})$ is said to be $M\mathbb{Z}$ shift invariant if $T_{Mz}W \subset W$ for all $z \in \mathbb{Z}$.

Given a frame wavelet Ψ, the space of negative dilates $V(\Psi)$ is:

$$V(\Psi) = \overline{\text{span}}\{\psi_{j,k} \,:\, j < 0, k \in \mathbb{Z}, \psi \in \Psi\}.$$

The local commutant of a system of operators $\mathcal{A}$ at the point $f \in L^2(\mathbb{R})$ is defined as

$$\mathcal{C}_f(\mathcal{A}) := \left\{B \in \mathcal{B}(L^2(\mathbb{R})) : BAf = ABf \quad \forall\, A \in \mathcal{A}\right\}.$$

The Canonical Dual Frame

The canonical dual of a wavelet frame $\{D_a^j T_k \psi\}_{j,k\in\mathbb{Z},\psi\in\Psi}$ is:

$$\begin{aligned}\{\mathrm{S}^{-1}\, D_a^j T_k \psi_i : j,k\in\mathbb{Z}, i=1,\dots,L\} &= \{D_a^j\, \mathrm{S}^{-1}\, T_k \psi_i : j,k\in\mathbb{Z}, i=1,\dots,L\} \\ &= \{D_a^j \eta^{k,i} : j,k\in\mathbb{Z}, i=1,\dots,L\},\end{aligned}$$

where S is the frame operator of $\mathcal{U}(\Psi)$, and $\{\eta^{k,i}\}$ is a family of functions, not necessarily with translation structure, indexed by $\{1,\dots,L\}\times\mathbb{Z}$.

The Canonical Dual Frame

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where S is the frame operator of $\mathcal{U}(\Psi)$, and $\{\eta^{k,i}\}$ is a family of functions, not necessarily with translation structure, indexed by $\{1,\dots,L\}\times\mathbb{Z}$. The canonical dual takes the form of a wavelet system generated by $|\Psi| = L$ functions precisely when

$$T_k \mathrm{S}^{-1} \psi = \mathrm{S}^{-1} T_k \psi \quad \text{for all } \psi\in\Psi, k\in\mathbb{Z},$$

that is, precisely when $\mathrm{S}^{-1} \in \mathcal{C}_\psi(\{T_k : k\in\mathbb{Z}\})$ for all $\psi\in\Psi$.

Lifting duality to a Sparser Lattice

For simplicity let $L = 1$, *i.e.*, $\Psi = \{\psi\}$. What can we "do" if $T_k \, S^{-1} \, \psi \neq S^{-1} \, T_k \psi$ for some $k \in \mathbb{Z}$, *i.e.*, , if the canonical dual is not a wavelet system generated by one function?

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Idea: For $P \in \mathbb{N}$ consider $\{D_a^j T_k \psi\}_{j,k \in \mathbb{Z}}$ as a wavelet system on the form $\{D_a^j T_{Pk} \psi\}_{j,k \in \mathbb{Z}, \psi \in \{\psi, T_1 \psi, \dots, T_{P-1} \psi\}}$.

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Now, it might happen that *this* system has a canonical dual with wavelet structure as systems on the sparser translation lattice $P\mathbb{Z}$ and P generators. This happens precisely when

$$T_{Pk} \,\mathrm{S}^{-1}\, \psi = \mathrm{S}^{-1}\, T_{Pk} \psi \quad \text{for all } k \in \mathbb{Z}.$$

We note that the frame operator S remains the same during this change of lattice.

Lifting duality to a Sparser Lattice

For simplicity let $L = 1$, *i.e.*, $\Psi = \{\psi\}$. What can we "do" if $T_k\, \mathsf{S}^{-1}\, \psi \neq \mathsf{S}^{-1}\, T_k \psi$ for some $k \in \mathbb{Z}$, *i.e.*, , if the canonical dual is not a wavelet system generated by one function?

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Then we have P functions $\{\phi_0, \phi_1, \ldots, \phi_{P-1}\}$ such that

$$\{D^j T_{Pk}(\phi_0), D^j T_{Pk}(\phi_1), \ldots, D^j T_{Pk}(\phi_{P-1})\}_{j,k\in\mathbb{Z}}$$

is the canonical dual of

$$\{D^j T_{Pk}(\psi), D^j T_{Pk}(T_1\psi), \ldots, D^j T_{Pk}(T_{P-1}\psi)\}_{j,k\in\mathbb{Z}}$$

The Period of a Frame Wavelet

Definition

Suppose that $\Psi = \{\psi_1, \ldots, \psi_L\} \subset L^2(\mathbb{R})$ is a frame wavelet associated with an integer dilation factor a, $|a| \geq 2$. The period of Ψ is the smallest integer $p \geq 1$ such that for all $f \in \overline{\text{span}}\,\{T_k\psi : k \in \mathbb{Z}, \psi \in \Psi\}$,

$$T_{pk}\, \mathrm{S}^{-1} f = \mathrm{S}^{-1}\, T_{pk} f \qquad \text{for all } k \in \mathbb{Z},$$

where S is the frame operator of the wavelet frame generated by Ψ. If there is no such p, we say that the period of Ψ is ∞.

Periods and the Canonical Dual

Proposition (Bownik, L.)

Suppose that $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2(\mathbb{R})$ *is a frame wavelet. For any nonnegative integer* $M \in \mathbb{N}$*, the following are equivalent:*

(i) $P(\Psi) \mid M$*, i.e., the period of* Ψ*, denoted* $P(\Psi)$*, divides* M*.*

(ii) There exist ML *functions* $\Phi = \{\phi_1, \dots, \phi_{ML}\}$ *such that* $\{D_a^j T_{Mk}\phi\}_{j,k\in\mathbb{Z},\phi\in\Phi}$ *is the canonical dual of* $\{D_a^j T_k\psi\}_{j,k\in\mathbb{Z},\psi\in\Psi} = \{D_a^j T_{Mk}\psi\}_{j,k\in\mathbb{Z},\psi\in\Psi_M}$*, where*

$$\Psi_M := \{T_m\psi : m = 0, \dots, M-1, \psi \in \Psi\}.$$

Hence, if the period $P(\Psi)$ of a frame wavelet Ψ is finite, then the canonical dual frame is a wavelet system generated by $P(\Psi) \cdot |\Psi|$ functions, and this is the least number of generators.

Periods and the Canonical Dual

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$$\Psi_M := \{T_m\psi : m = 0, \ldots, M-1, \psi \in \Psi\}.$$

Note: In this case the wavelet structure of the canonical dual frame is altered since it is based on the translation lattice $P(\Psi)\cdot\mathbb{Z}$ which is sparser than the original lattice $\mathbb{Z}$.

The Main Result (again)

Theorem

For all $J \in \mathbb{N}$, there exists a frame wavelet $\psi \in L^2(\mathbb{R})$ such that:

(i) *$\hat{\psi}$ is C^∞ and compactly supported,*

(ii) *its canonical dual frame is not a wavelet system generated by fewer than 2^J functions,*

(iii) *there are infinitely many $\tilde{\psi}$ such that ψ and $\tilde{\psi}$ form a pair of dual wavelet frames.*

Idea of the Proof (case $J = 1$)

Construct a nice frame wavelet with an alternate wavelet dual and a non $\mathbb{Z}$ shift invariant space of negative dilates $V(\psi)$. Use the negation of the result below to conclude that the canonical dual cannot be generated by one function.

Proposition (Bownik & Weber)

Suppose that the canonical dual of a wavelet frame ψ has a wavelet structure, i.e., it is of the form $\{\phi_{j,k} : j, k \in \mathbb{Z}\}$ for some frame wavelet ϕ. Then, the space of negative dilates

$$V(\psi) = \overline{\operatorname{span}}\{\psi_{j,k} : j < 0,\ k \in \mathbb{Z}\}$$

is shift-invariant.

Sketch of the Proof (general case $J \in \mathbb{N}$)

Fix $J \in \mathbb{N}$. Construct a smooth frame wavelet $\psi = \psi^0 + \varepsilon\psi^1$ as in the first lemma and the figure with $N = J + 3$ and with a space of negative dilates $V(\psi)$ not shift invariant under $M\mathbb{Z}$ for any $M = 1, 2, \ldots, 2^J$ (second lemma).

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Proposition (Bownik & Weber)

Let $M \in \mathbb{N}$. If Ψ is a frame wavelet and the period of Ψ divides M, then $V(\Psi)$ is shift invariant by the lattice $M\mathbb{Z}$.

First lemma: Constructing the Frame Wavelet

Lemma

For every $N \geq 4$ and $0 < \delta < 2^{-N}$, there exists a frame wavelet ψ such that $\hat{\psi} \in C_0^\infty(\mathbb{R})$ and

$$\begin{aligned} \hat{\psi}(\xi) \neq 0 \iff \xi \in & (-1/2, -1/4) \cup (1/2, 3/4) \\ & \cup \left(-2^{-N+1} - \delta, -2^{-N} + \delta\right) \\ & \cup \left(2^{-N} - \delta, 2^{-N+1} + \delta\right) \end{aligned}$$

$$\hat{\psi}(\xi) = \hat{\psi}(\xi - 1) \neq 0 \qquad \text{for } \xi \in (1/2, 3/4)\,.$$

First lemma: Constructing the Frame Wavelet

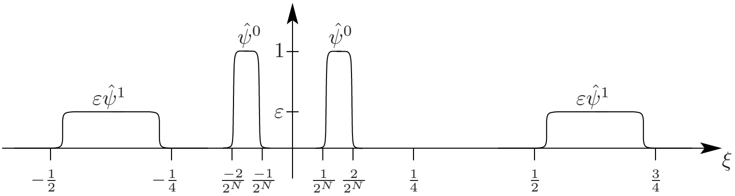

Figure: Sketch of the graph of $\hat{\psi} = \hat{\psi}^0 + \varepsilon\hat{\psi}^1$; ψ and ψ^0 are (alternate) duals.

Second Lemma: Showing Non Shift Invariance of $V(\psi)$

Lemma

Suppose that $\psi \in L^2(\mathbb{R})$ is as in the first lemma with $N \geq 4$ and $0 < \delta < 2^{-N}$. Then the space of negative dilates $V(\psi)$ is not $p\mathbb{Z}$-SI for any $p < 2^{N-3}$, $p \in \mathbb{N}$.

Sketch of proof.

Let $W_j := \overline{\text{span}}\{\psi_{j,k} : k \in \mathbb{Z}, \psi \in \Psi\}$ for each $j \in \mathbb{Z}$. Use that $V(\psi) = \overline{\text{span}} \bigcup_{j<0} W_j(\psi)$, that $f \in W_0(\psi)$ if and only if $f(2^j \cdot) \in W_j(\psi)$, and that the principal shift-invariant subspace $W_0(\psi)$ can be described as

$$W_0(\psi) = \{f \in L^2(\mathbb{R}) : \hat{f} = \hat{\psi} m \quad \text{for some meas., 1-periodic } m\}.$$

$\square$

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$$W_0(\psi) = \Big\{ f \in L^2(\mathbb{R}) : \text{supp}\, \hat{f} \subset [-1/2, -1/4] \cup [1/2, 3/4] \cup K \\ \hat{f}(\xi - 1) = \hat{f}(\xi) \quad a.e.\, \xi \in [1/2, 3/4] \Big\},$$

where $K = \left[-2^{-N+1} - \delta, -2^{-N} + \delta\right] \cup \left[2^{-N} - \delta, 2^{-N+1} + \delta\right]$.

Second Lemma: Showing Non Shift Invariance of $V(\psi)$

Proof (cont.)

$$
\begin{aligned}
V(\psi) = \Big\{ f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subset \bigcup_{j=-N+3}^{-1} 2^j([-1/2,-1/4] \cup [1/2,3/4]),& \\
\hat{f}(\xi - 2^{-1}) = \hat{f}(\xi) \quad a.e.\, \xi \in \left[2^{-2}, 3/2 \cdot 2^{-2}\right],& \\
\hat{f}(\xi - 2^{-2}) = \hat{f}(\xi) \quad a.e.\, \xi \in \left[2^{-3}, 3/2 \cdot 2^{-3}\right],& \\
\vdots \qquad\qquad\qquad \vdots \qquad& \\
\hat{f}(\xi - 2^{-N+3}) = \hat{f}(\xi) \quad a.e.\, \xi \in \left[2^{-N+2}, 3/2 \cdot 2^{-N+2}\right] \Big\}.&
\end{aligned}
$$

Define $f \in V(\psi)$ by $\hat{f} = \chi_{[2^{-N+2}, 3/2 \cdot 2^{-N+2}] \cup [-2^{-N+2}, -1/2 \cdot 2^{-N+2}]}$. Show that $T_p f \notin V(\psi)$ for $p < 2^{N-3}$, or equivalently, that $\exp(2\pi i p \cdot)\hat{f} \notin \mathcal{F}V(\psi)$. $\square$

In Perspective with Other Known Results

Theorem (Aucher, 1990)

Suppose $\psi \in L^2(\mathbb{R})$ is an orthonormal wavelet, $|\hat{\psi}|$ continuous, and $\hat{\psi} = \mathcal{O}(|\xi|^{-1/2-\delta})$ for some $\delta > 0$. Then ψ is an MRA wavelet.

Hence, all "regular" orthonormal wavelets are associated with an MRA.

In Perspective with Other Known Results

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Hence, all "regular" orthonormal wavelets are associated with an MRA. Baggett et al. constructed a non-MRA C^r tight frame wavelet with rapid decay for any $r \in \mathbb{N}$. The non-tight frame wavelet from the main result is an example of a non-GMRA C^∞ frame wavelet with rapid decay.

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The Frame Operator

Proposition

Suppose that $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2(\mathbb{R})$ generates a wavelet system which is a Bessel sequence. Let

$$\mathcal{D} = \{f \in L^2(\mathbb{R}) : \hat{f} \in L^\infty(\mathbb{R}) \text{ and } \operatorname{supp} \hat{f} \subset [-R, -1/R] \cup [1/R, R] \text{ for } R > 1\}.$$

Then its frame operator S *is given by*

$$\widehat{\mathrm{S}\, f}(\xi) = \hat{f}(\xi) \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} |\hat{\psi}_l(2^j \xi)|^2 + \sum_{p\in\mathbb{Z}} \sum_{q\in 2\mathbb{Z}+1} \hat{f}(\xi + 2^{-p} q) t_q(2^p \xi)$$

for a.e. $\xi \in \mathbb{R}$ and for all $f \in \mathcal{D}$, where

$$t_q(\xi) = \sum_{l=1}^{L} \sum_{j=0}^{\infty} \hat{\psi}_l(2^j \xi) \overline{\hat{\psi}_l(2^j(\xi + q))} \qquad \text{for } q \in \mathbb{Z}.$$

This proposition is implicitly contained in the book of Hernández and Weiss, Proposition 7.1.19.

Lifting

We are in some sense lifting the duality to the translation lattice $P\mathbb{Z}$ and paring

$$\{D^j T_{Pk}(\psi), D^j T_{Pk}(T_1\psi), \ldots, D^j T_{Pk}(T_{P-1}\psi)\}_{j,k\in\mathbb{Z}}$$

with

$$\{D^j T_{Pk}(\phi_0), D^j T_{Pk}(\phi_1), \ldots, D^j T_{Pk}(\phi_{P-1})\}_{j,k\in\mathbb{Z}},$$

or, equivalently, lifting the duality to scale m and paring

$$\{D^j T_k(D^m\psi), D^j T_k(D^m T_1\psi), \ldots, D^j T_k(D^m T_{P-1}\psi)\}_{j,k\in\mathbb{Z}}$$

with

$$\{D^j T_k(\tilde{\phi}_0), D^j T_k(\tilde{\phi}_1), \ldots, D^j T_k(\tilde{\phi}_{P-1})\}_{j,k\in\mathbb{Z}},$$

where $\tilde{\phi}_i = D^m\phi_i$ for $i \in \{0, 1, \ldots, P-1\}$. We note that the frame operator S remains the same during this lifting.